Thermal QCD sum rules for mesons

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A recently proposed scheme is used to saturate the spectral side of the QCD sum rules derived from the *thermal*, two-point correlation functions of the vector and the axial-vector currents. At low temperature, it constructs the spectral representation from all the one-loop Feynman diagrams for the two-point functions. The old saturation scheme treats incorrectly some of these contributions. We end up with the familiar QCD sum rules obtained from the difference of the two corresponding *vacuum* correlation functions. The possibility of obtaining new sum rules in other media is discussed.

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I. INTRODUCTION

In their original work extending the vacuum QCD sum rules [1] to those at finite temperature, Bochkarev and Shaposhnikov [2] recognized the importance of multiparticle (branch cuts) contributions in addition to those of single particles (poles) in constructing the spectral representation of thermal two-point functions. Thus for the vector current correlation function, they included not only the ρ pole with temperature-dependent residue and position, but also the $\pi\pi$ continuum. In the same way the spectral representation for the nucleon current correlation function would consist of the nucleon pole with temperature-dependent parameters and the πN continuum. Although such a saturation scheme is quite suggestive and had been extensively used in the past [3,4], it lacks a theoretical basis, leaving one to suspect if some equally important contributions are left out.

A definitive saturation scheme emerged with the work of Leutwyler and Smilga [5], who calculated the nucleon current correlation function in chiral perturbation theory. Being interested in the shifts of the nucleon pole parameters at low temperature, they considered all the one-loop Feynman diagrams for the correlation function and evaluated them in the vicinity of the nucleon pole. Koike [6] examined the contributions of these diagrams in the context of QCD sum rules. He found a new contribution arising from the nucleon self-energy diagram to the sum rules, not required in the saturation scheme mentioned above.

The one-loop Feynman diagrams for the nucleon correlation function were further analyzed in Ref. [7]. In this set of diagrams for the correlation function, one has not only diagrams with the (single particle) pole alone and the (two particle) branch points alone, but also other (one particle reducible) diagrams, appearing as a product of factors with the pole and the branch cut. As an example, take the case of a vertex correction diagram having this product structure. It may be expressed as the sum of the pole term with constant residue and a remainder, regular at the pole. Clearly to find the pole parameters one may confine oneself to the pole term alone. But if one wishes to evaluate the diagram for large spacelike momenta—the region of relevance for the QCD sum rules—both the pole and the remainder become of comparable magnitude. It is these remainder terms which are not included in the earlier saturation scheme.

In this paper we write down the thermal QCD sum rules following from the vector current and the axial vector current correlation functions, constructing the spectral side from the set of all one-loop Feynman diagrams. At low temperature only the distribution function of the pions is significant in the heat bath. Thus, of the two particles in the loop, at least one must be a pion, the other being any one of the strongly interacting particles with appropriate quantum numbers in the low mass region. The vertices occurring in the diagrams can be obtained from the chiral symmetry of QCD alone [8,9], as well as from other related formulations, assuming additional symmetries [10,11]. *A priori*, the Lagrangian with the chiral symmetry alone is singled out, if we claim the sum rules to follow from QCD.

As with the nucleon sum rules [7], we subtract out the vacuum sum rules from the corresponding full sum rules at finite temperature, equating, in effect, terms of $O(T^2)$ on both sides. All our calculations are done in the chiral symmetry limit, though we keep nonvanishing pion mass in intermediate steps.

In Sec. II we collect some results to be used later. In Sec. III we analyze the one-loop Feynman diagrams to construct the spectral representation for the correlation functions. In Sec. IV we use the known results of operator product expansion and write the sum rules. Finally our concluding remarks are contained in Sec. V.

II. PRELIMINARIES

Here we review briefly the kinematics of the two point correlation functions of the vector and the axial-vector currents. Then we write the leading interaction vertices allowed by the chiral symmetry of QCD and calculate the loop integrals we shall meet in our work.

A. Kinematics

Consider the two thermal correlation functions,

$$T^{ab}_{\mu\nu}=i\int d^4x e^{iq\cdot x}\mathrm{Tr}\varrho TV^a_\mu(x)V^b_\nu(0) \tag{2.1}$$

and

$$T'^{ab}_{\mu\nu}=i\int d^4x e^{iq\cdot x}\mathrm{Tr}\varrho T A^a_\mu(x)A^b_\nu(0) \tag{2.2}$$

of the vector and the axial vector currents,

$$V^a_\mu(x)=\overline{q}(x)\gamma_\mu\frac{\tau^a}{2}q(x),\quad A^a_\mu(x)=\overline{q}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}q(x),$$

generated by the $SU(2)$ flavor symmetry group of the QCD Lagrangian. Here τ^a are the Pauli matrices and $\varrho=e^{-\beta H}/\mathrm{Tr}e^{-\beta H}$ is the thermal density matrix of QCD at temperature $T=1/\beta$. Note that in the limit of chiral symmetry, in which we shall work, the axial vector current is also conserved and the kinematics, in particular, the invariant decomposition is the same for both the correlation functions.

The current conservation leads to the invariant decomposition

$$T^{ab}_{\mu\nu}(q)=\delta^{ab}(P_{\mu\nu}T_t+Q_{\mu\nu}T_l), \tag{2.3}$$

where the gauge invariant tensors are chosen as

$$P_{\mu\nu}=-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}-\frac{q^2}{\overline{q}^2}\widetilde{u}_\mu\widetilde{u}_\nu,\quad Q_{\mu\nu}=\frac{q^4}{\overline{q}^2}\widetilde{u}_\mu\widetilde{u}_\nu,$$

with $\widetilde{u}_\mu=u_\mu-\omega q_\mu/q^2$, where u_μ is the four-velocity of the medium and ω and $\overline{q}$ are Lorentz invariant scalars, $\omega=u\cdot q$ and $\overline{q}=\sqrt{\omega^2-q^2}$, representing the time and space components of q_μ in the rest frame of the heat bath ($u_0=1$, $\vec{u}=0$). The invariant amplitudes are functions of the scalar variables, say, q^2 and ω. They can be conveniently extracted from the Feynman diagrams by forming the scalars,

$$T_1=g^{\mu\nu}T_{\mu\nu},\quad T_2=u^\mu u^\nu T_{\mu\nu}, \tag{2.4}$$

which are simply related to the invariant amplitudes.

Having cast the kinematics in a Lorentz invariant form, we choose to do calculations in the rest frame of the heat bath. The kinematic decomposition (2.3) leads to a constraint on the invariant amplitudes which in this frame reads as

$$T_t(q_0,\vec{q}=0)=q_0^2T_l(q_0,\vec{q}=0). \tag{2.5}$$

Using this equation, the two sets of amplitudes $T_{l,t}$ and $T_{1,2}$ can be related for $\vec{q}=0$ as

$$T_l=\frac{1}{\overline{q}^2}T_2,\quad T_t=-\frac{1}{3}T_1. \tag{2.6}$$

Note also the symmetry of the imaginary parts of the amplitudes,

$$\mathrm{Im}T_{l,t}(-q_0,\vec{q}=0)=\mathrm{Im}T_{l,t}(q_0,\vec{q}=0). \tag{2.7}$$

In the real time thermal field theory that we are going to use here, each of the above amplitudes stands for a 2×2 matrix, whose components depend on a single analytic function [12]. This function, in turn, is determined completely by the 11-component function itself: Their real parts are equal and the imaginary part of the former equals $\pi^{-1}\tanh(\beta q_0/2)$ times that of the latter. (The factor π^{-1} is inserted for convenience.) To avoid further symbols, we henceforth redefine T to denote this analytic function. Its spectral representation at fixed $\vec{q}$ is given by

$$T_{l,t}(q_0^2,\vec{q})=\int_0^\infty\frac{dq_0'^2\,\mathrm{Im}T_{l,t}(q_0',\vec{q})}{q_0'^2-q_0^2-i\epsilon}. \tag{2.8}$$

B. Dynamics

Here we write down the chirally symmetric effective Lagrangian involving pions and spin one mesons. To calculate the correlation functions we also introduce external fields $v^a_\mu(x)$ and $a^a_\mu(x)$ coupled to the currents $V^a_\mu(x)$ and $A^a_\mu(x)$ [8], extending the original QCD Lagrangian as

$$\mathcal{L}_{QCD}\to\mathcal{L}_{QCD}+v^a_\mu(x)V^\mu_a(x)+a^a_\mu(x)A^\mu_a(x).$$

The resulting interaction vertices are obtained in Refs. [13,14] for the chiral symmetry group $SU(3)_R\times SU(3)_L$.

At low temperature the pions dominate the heat bath. We thus consider the reduced symmetry $SU(2)_R\times SU(2)_L$ and write down the relevant pieces of the chiral couplings of pions with themselves, the external fields and the other observed particles, to be encountered in the one-loop diagrams [15]. These are given by

$$\mathcal{L}_{int}(\pi,v,a)=\mathcal{L}(\pi)+\mathcal{L}_v(\pi)+\mathcal{L}_a(\pi) \tag{2.9}$$

where

$$\mathcal{L}(\pi)=-\frac{1}{6F_\pi^2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}-\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}),$$

$$\mathcal{L}_v(\pi)=\boldsymbol{v}_\mu\cdot\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi}+\frac{1}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\boldsymbol{v}_\mu\cdot\boldsymbol{v}^\mu-\boldsymbol{v}_\mu\cdot\boldsymbol{\pi}\boldsymbol{v}^\mu\cdot\boldsymbol{\pi}), \tag{2.10}$$

$$\mathcal{L}_a(\pi)=\frac{1}{2}F_\pi^2\boldsymbol{a}_\mu\cdot\boldsymbol{a}^\mu-F_\pi\boldsymbol{a}_\mu\cdot\partial^\mu\boldsymbol{\pi}-\frac{1}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\boldsymbol{a}_\mu\cdot\boldsymbol{a}^\mu-\boldsymbol{a}_\mu\cdot\boldsymbol{\pi}\boldsymbol{a}^\mu\cdot\boldsymbol{\pi})+\frac{2}{3F_\pi}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}\cdot\boldsymbol{a}_\mu-\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\boldsymbol{\pi}\cdot\boldsymbol{a}_\mu),$$

the letters in boldface denoting isospin vectors. Since we are considering thermal corrections to one loop only, we may already evaluate the single pion loops generated by the above interaction vertices. Such a loop is given by the 11-component of the thermal pion propagator formed by contracting the two pion fields without derivatives in such vertices,

$$\Delta_{11}^{(\pi)}(k)=\frac{i}{k^2-m_\pi^2}+2\pi\delta(k^2-m_\pi^2)n(k), \qquad (2.11)$$

where $n(k_0)=(e^{\beta|k_0|}-1)^{-1}$ is the Bose distribution function. Then the thermal part of the loop is given by

$$\mathrm{Tr}\varrho T\pi^a(x)\pi^b(x)|_{11}\to\delta^{ab}\int\frac{d^4k}{(2\pi)^3}n(k_0)\delta(k^2-m_\pi^2)$$
$$=\delta^{ab}\frac{T^2}{12}, \qquad (2.12)$$

in the chiral limit. Thus we can write $\mathcal{L}(\pi)$ and $\mathcal{L}_a(\pi)$ as the effective two-point vertices,

$$\mathcal{L}(\pi)=-\frac{T^2}{36F_\pi^2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi},$$

$$\mathcal{L}_a(\pi)=\frac{F_\pi^2}{2}\left(1-\frac{T^2}{6F_\pi^2}\right)\boldsymbol{a}_\mu\cdot\boldsymbol{a}^\mu-F_\pi\left(1-\frac{T^2}{9F_\pi^2}\right)\boldsymbol{a}_\mu\cdot\partial^\mu\boldsymbol{\pi}. \qquad (2.13)$$

Next, we write the couplings of the isotriplets [$\rho(770)$, $a_1(1230)$] and isosinglets [$\omega(782)$, $f_1(1282)$] of vector (1^{--}) and axial vector (1^{++}) mesons, respectively. Although we take their fields to transform according to $SU(2)$ in constructing their interactions, we take the physical (zero temperature) masses of the multiplets of each of the $SU(3)$ octets to be degenerate. Then the couplings linear in the vector meson fields are given by

$$\mathcal{L}(V)=\frac{F_\rho}{m_V}\left\{\left(1-\frac{T^2}{12F_\pi^2}\right)\partial^\mu\boldsymbol{v}^\nu\cdot(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)+\frac{1}{F_\pi}\partial^\mu\boldsymbol{a}^\nu\cdot(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)\times\boldsymbol{\pi}\right\}$$
$$-\frac{2G_\rho}{m_VF_\pi^2}\partial_\mu\boldsymbol{\rho}_\nu\cdot\partial^\mu\boldsymbol{\pi}\times\partial^\nu\boldsymbol{\pi}$$
$$-\frac{\sqrt{2}H_\omega}{m_VF_\pi}\epsilon_{\mu\nu\lambda\sigma}\omega^\mu\partial^\nu\boldsymbol{\pi}\cdot\partial^\lambda\boldsymbol{v}^\sigma, \qquad (2.14)$$

while those linear in the axial vector meson fields are

$$\mathcal{L}(A)=-\frac{F_{a_1}}{m_A}\left\{\left(1-\frac{T^2}{12F_\pi^2}\right)\partial^\mu\boldsymbol{a}^\nu\cdot(\partial_\mu\boldsymbol{a}_{1\nu}-\partial_\nu\boldsymbol{a}_{1\mu})+\frac{1}{F_\pi}\partial^\mu\boldsymbol{v}^\nu\cdot(\partial_\mu\boldsymbol{a}_{1\nu}-\partial_\nu\boldsymbol{a}_{1\mu})\times\boldsymbol{\pi}\right\}$$
$$+\frac{\sqrt{2}H_{f_1}}{m_AF_\pi}\epsilon_{\mu\nu\lambda\sigma}f_1^\mu\partial^\nu\boldsymbol{\pi}\cdot\partial^\lambda\boldsymbol{a}^\sigma, \qquad (2.15)$$

where we have again contracted the two pion fields at the vertices forming single pion loops. Finally the quadratic couplings of the triplets with the singlets and between themselves are given by [15],

$$\mathcal{L}(V,A)=-2\epsilon_{\mu\nu\lambda\sigma}\left(\frac{g_1}{F_\pi}\partial^\mu\omega^\nu\boldsymbol{\rho}^\lambda\cdot\partial^\sigma\boldsymbol{\pi}+\frac{g_2}{F_\pi}\partial^\mu f_1^\nu\boldsymbol{a}_1^\lambda\cdot\partial^\sigma\boldsymbol{\pi}\right)$$
$$+\frac{g_3}{F_\pi}\partial^\mu\boldsymbol{\rho}^\nu\cdot(\boldsymbol{a}_{1\mu}\times\partial_\nu\boldsymbol{\pi}-\boldsymbol{a}_{1\nu}\times\partial_\mu\boldsymbol{\pi}). \qquad (2.16)$$

The coupling constants in the above interaction terms can be determined from the decay rates of the particles [14,15].

There also appear two other spin-one mesons $h_1(1170)$ and $b_1(1235)$ having quantum numbers 1^{+-} [16]. But P and C invariance forbids their appearance in our diagrams [13]. The pseudoscalar $SU(2)$ singlets $\eta(547)$ and $\eta'(985)$ also have no couplings relevant for us. The scalars $\sigma(400-1200)$ and $a_0(980)$ do have vertices for our diagrams, but they do not contribute to $O(T^2)$, as do many of the vertices written above. We shall omit the vertices with the scalars altogether from our discussions to follow.

We comment here on the possibility of constructing a full-fledged heavy meson chiral perturbation theory with an effective Lagrangian consisting of a string of terms with increasing number of derivatives (and quark mass factors), Eqs. (2.10), (2.14)–(2.16) representing the leading term. It would then yield all (vacuum) Green's functions as series in powers of momenta, while the series for thermal averages would be in powers of temperature. The difficulty here lies in the existence of vertices like $\rho\pi\pi$, leading to ρ decaying into two non-soft pions. It signals a possible breakdown of the chiral perturbation expansion; at least, it questions the evaluation of a quantity by its leading term.

However, this lack of a proper chiral perturbation theory involving the vector and the axial vector mesons does not concern us here. For the evaluation of the sum rules to the leading order in temperature, it suffices to find the leading terms of different quantities in the sum rules, even though these terms alone do not represent their reliable estimates.

It is interesting to compare the Lagrangian written above with the other well-known chiral Lagrangian involving the vector and the axial-vector mesons, namely the one with massive Yang-Mills or hidden-gauge bosons [10,11]. It turns out that such a formulation gives essentially the same terms as we have obtained by imposing chiral symmetry alone, only some of their couplings get related [14]. Since these relations are satisfied well by experimental data, this formulation must be considered the same as ours.

C. Loop integrals

With the above interaction vertices we can write the Feynman amplitudes for the correlation functions. A loop in the diagrams makes a contribution of the form

$$F_{\mu\nu}(q)=ic\int\frac{d^4k}{(2\pi)^4}f_{\mu\nu}(q,k)\Delta_{11}^{(\pi)}(k)\Delta_{11}^{(X)}(q-k), \qquad (2.17)$$

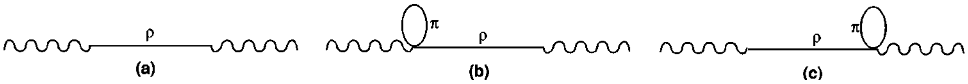

FIG. 1. ρ pole and constant vertex correction.

where the particle X can be a heavy (spin one) meson or the pion itself. The tensor $f_{\mu\nu}$ is given by the interaction vertices and any tensor structure in $\Delta_{11}^{(X)}$. Out of the interaction vertices we have isolated the coupling constants in c. Being gauge invariant, $F_{\mu\nu}$ allows one to construct the invariant amplitudes $F_{l,t}$ in the same way as we did above for $T_{\mu\nu}$.

As we shall see below, the one-particle irreducible diagrams consist only of loop integrals like (2.17), while the reducible ones are given by such integrals multiplied with a heavy particle pole (of first or second order). In the former case we may evaluate directly the leading contribution of the integral and then Borel transform. In the latter case it is more systematic to use the exact spectral representation of the loop integral. We then Borel transform the resulting complete amplitude and extract its leading term.

Until now we have considered full amplitudes, including the corresponding vacuum amplitudes. Our thermal sum rules need only their temperature-dependent parts. So we evaluate only these (convergent) parts of the loop integrals, but continue to denote them by the same symbols as for the full amplitudes.

Consider first the case where X is a heavy particle of mass m_H, whose propagator is well approximated by the one in vacuum. Let the full amplitude be given by just the loop integral (2.17). Going over to the invariant amplitudes, its thermal part is as

$$F_{l,t}(q)=-c\int\frac{d^4k}{(2\pi)^3}\frac{\delta(k^2-m_\pi^2)n(k)}{(q-k)^2-m_H^2}f_{l,t}(q,k), \tag{2.18}$$

whose leading behavior for large spacelike momenta with $\vec{q}=0$ $(q_0^2=E^2=-Q^2<0)$ is given by $(\omega_k=\sqrt{\vec{k}^2+m_\pi^2})$

$$F_{l,t}(Q^2,\vec{q}=0)\rightarrow\frac{2c}{Q^2+m_H^2}\int\frac{d^3kn(k)}{(2\pi)^3 2\omega_k}f_{l,t}(Q,|\vec{k}|). \tag{2.19}$$

It is clear that only if $f_{l,t}(Q,|\vec{k}|)$ is constant in $\vec{k}$, is the integral of order T^2; otherwise, it is of higher order.

To obtain the spectral representation of the loop integral for $\vec{q}=0$, we note that the cuts in the E^2 plane in this case are given by $0<E^2<(m_H-m_\pi)^2$ and $E^2>(m_H+m_\pi)^2$. The imaginary parts across both these cuts are given by [7]

$$\mathrm{Im}F_{l,t}(E)=\frac{\sqrt{\omega^2-m_\pi^2}}{8\pi^2E}n(\omega)f_{l,t}(E,\omega) \tag{2.20}$$

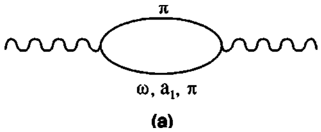

FIG. 2. Intermediate state diagrams.

where $\omega=(E^2-m_H^2+m_\pi^2)/2E$. We now have the spectral representation for $F_{l,t}$ given by an equation like (2.8), the integration running over the two cuts stated above.

If the particle X is also a pion, the leading term of the loop integral may again be found in the same way as we did above for the heavy particle. However, to find the spectral representation for $\vec{q}=0$, one must begin with $\vec{q}\neq 0$ and then go to the limit. This evaluation is done in the Appendix.

III. SPECTRAL REPRESENTATION

Here we analyze the different one-loop Feynman diagrams for the correlation functions and extract the leading thermal contributions (to order T^2) to the spectral side of the sum rules. These diagrams are the same as those considered in [15] to find the shifts in the pole parameters of the ρ and a_1 mesons. The difference lies in the evaluation of the diagrams: While we evaluated them earlier in the neighborhood of the respective poles, now we have to do so at large spacelike momenta.

The diagrams for the correlation functions can be grouped into three types, namely those with intermediate states, vertex corrections and self-energies. To write the Feynman amplitudes we anticipate the following gauge invariant tensors:

$$A_{\mu\nu}(q)=-g_{\mu\nu}+q_\mu q_\nu/q^2=P_{\mu\nu}+Q_{\mu\nu}/q^2,$$

$$B_{\mu\nu}(q,k)=q^2k_\mu k_\nu-q\cdot k(q_\mu k_\nu+k_\mu q_\nu)+(q\cdot k)^2g_{\mu\nu}, \tag{3.1}$$

$$C_{\mu\nu}(q,k)=q^4k_\mu k_\nu-q^2(q\cdot k)(q_\mu k_\nu+k_\mu q_\nu)+(q\cdot k)^2q_\mu q_\nu.$$

We now consider the diagrams separately for the correlation functions of the vector and the axial vector currents. Although we need only the T-dependent parts of the diagrams to order T^2, we shall write the pole amplitudes in full.

A. Vector current

Here the pole and the one-loop diagrams are shown in Figs. 1–4.

The ρ pole amplitude with its constant vertex corrections (Fig. 1) is given by

$$T_{\mu\nu}^{(\rho)}(q)=-\left(\frac{F_\rho}{m_V}\right)^2\left(1-\frac{T^2}{6F_\pi^2}\right)\frac{q^4}{q^2-m_V^2}A_{\mu\nu}. \tag{3.2}$$

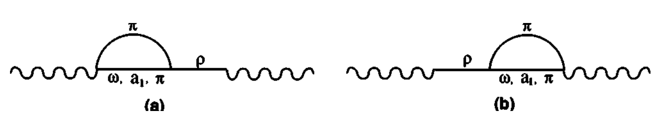

FIG. 3. Vertex correction diagrams.

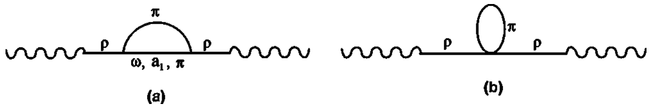

FIG. 4. Self-energy diagrams.

The amplitudes for the intermediate state diagrams (Fig. 2) are generally of the form of Eq. (2.17). In the case of the $\pi\pi$ intermediate state, one has to include also the tadpole diagram of Fig. 2(b) giving

$$T^{(\pi\pi)}_{\mu\nu}(q)=i\int\frac{d^4k}{(2\pi)^4}\{(2k-q)_\mu(2k-q)_\nu\Delta^{(\pi)}_{11}(k)\times\Delta^{(\pi)}_{11}(q-k)-2ig_{\mu\nu}\Delta^{(\pi)}_{11}(k)\},\tag{3.3}$$

whose thermal part is

$$T^{(\pi\pi)}_{\mu\nu}(q)=A_{\mu\nu}(-2q^2)\int\frac{d^4k}{(2\pi)^3}\frac{\delta(k^2-m_\pi^2)n(k)}{(q-k)^2-m_\pi^2}\to-\frac{T^2}{6}A_{\mu\nu},\tag{3.4}$$

at large spacelike momenta. For the πa_1 intermediate state, the amplitude is given by Eq. (2.17) with

$$c=2(F_{a_1}/m_AF_\pi)^2,$$

$$f_{\mu\nu}=q^2(q^2-2q\cdot k)A_{\mu\nu}-B_{\mu\nu},$$

getting

$$f_{l,t}=E^2(1,\ E^2).$$

Then Eq. (2.19) gives

$$T^{(\pi a_1)}_{l,t}\to\left(\frac{F_{a_1}}{m_A}\right)^2\frac{(-Q^2,Q^4)}{Q^2+m_A^2}\cdot\frac{T^2}{6F_\pi^2}.\tag{3.5}$$

The (Borel transforms of the) amplitudes given by Eqs. (3.2),(3.4),(3.5) constitute all of the contributions to the spectral side of the sum rules. In the rest of this section we verify that none of the remaining diagrams with $\pi\omega$ intermediate state, vertex corrections and self-energies contribute to order T^2 to the sum rules. Clearly it suffices to show this behavior for any one of the invariant amplitudes, say T_l, and we omit its subscript in the following.

For the $\pi\omega$ intermediate state, the amplitude is again given by Eq. (2.17) with

$$c=-2(H_\omega/m_VF_\pi)^2,\quad f_{\mu\nu}=q^2k^2A_{\mu\nu}+B_{\mu\nu},$$

$$f=-2|\vec{k}|^2/3,\tag{3.6}$$

where and below f stands for f_l. Then Eq. (2.19) shows immediately that this amplitude is of order T^4.

Considering next the vertex corrections of Fig. 3, each of the amplitudes is of the form

$$\frac{F_\rho}{m_V}\frac{q^2}{q^2-m_V^2}\Lambda_{\mu\nu},\tag{3.7}$$

where $\Lambda_{\mu\nu}$ is again a one-loop integral of the form of Eq. (2.17). Consider now

$$\frac{E^2}{E^2-m_V^2}\Lambda(E^2),\tag{3.8}$$

where the invariant amplitude $\Lambda(E^2)$ satisfies the spectral representation

$$\Lambda(E^2)=\int\frac{\mathrm{Im}\Lambda(E')dE'^2}{E'^2-E^2},$$

which may be used to write the expression (3.8) as

$$\frac{m_V^2\Lambda(m_V^2)}{E^2-m_V^2}+\int\frac{dE'^2E'^2\mathrm{Im}\Lambda(E')}{(E'^2-m_V^2)(E'^2-E^2)},\tag{3.9}$$

separating the pole term from the one regular in the vicinity of the pole. If we now go to large spacelike $E^2=-Q^2$ and take the Borel transform, it becomes

$$\frac{e^{-m_V^2/M^2}}{M^2}\int\frac{dE^2\mathrm{Im}\Lambda(E)}{(E^2-m_V^2)}(-m_V^2+E^2e^{-(E^2-m_V^2)/M^2}),\tag{3.10}$$

where M^2 is the Borel variable replacing Q^2.

Equations (3.9) and (3.10) allow us to compare how the same amplitude behaves near the pole and at large spacelike momenta in the form of the Borel transform. Consider the vertex correction from the $\pi\omega$ loop for which $c=-(4\sqrt{2}H_\omega g_1/m_VF_\pi^2)$ and the expressions for $f_{\mu\nu}$ and f are identical to those in Eqs. (3.6). Then we see that in Eq. (3.9) $\Lambda(m_V^2)\sim O(T^4)$ and the second term is indeed finite at $E^2=m_V^2$ (and of order T^2). Next consider the behavior of the Borel transform (3.10). Because the branch point of the function $\Lambda(E^2)$ also starts at $E^2=m_V^2$, its leading term is given by

$$\left(1-\frac{m_V^2}{M^2}\right)\frac{e^{-m_V^2/M^2}}{M^2}\int dE^2\mathrm{Im}\Lambda(E),\tag{3.11}$$

which is of order T^4. Thus to $O(T^2)$, the vertex correction diagram with the $\pi\omega$ loop contributes neither to the pole parameters nor to the Borel transform. Notice, however, that while near the pole the second term in Eq. (3.9) can be ignored, for large spacelike momenta both the terms assume equal importance.

Consider next the πa_1 loop with

$$c=-(4F_{a_1}g_3/m_AF_\pi^2),$$

$$f_{\mu\nu}=q^2q\cdot kA_{\mu\nu}+B_{\mu\nu},\quad f=E|\vec{k}|.$$

FIG. 5. π pole and constant vertex corrections.

Again we see that at $E^2=m_V^2$, the residue $\Lambda(m_V^2)\sim O(T^4)$. Here the branch point of $\Lambda(E^2)$ is at $E^2=m_A^2$. So the Borel transform is clearly of order T^4. As shown in the Appendix, the $\pi\pi$ loop contribution is also at least of order T^4 to the pole residue and Borel transform.

Finally the self-energy diagrams of Fig. 4 can also be analyzed in a similar way. Each of the diagrams contributes an amplitude of the form

$$-\left(\frac{F_\rho}{m_V}\right)^2\cdot\frac{q^4}{(q^2-m_V^2)^2}\Pi_{\mu\nu}(q). \tag{3.12}$$

The self-energy $\Pi_{\mu\nu}$ is again of the form of Eq. (2.17). Using the spectral representation for its invariant amplitudes, we get

$$\begin{aligned}\frac{E^4}{(E^2-m_V^2)^2}\Pi(E)=&\frac{m_V^4}{(E^2-m_V^2)^2}\int\frac{dE'^2\mathrm{Im}\Pi(E')}{E'^2-m_V^2}+\frac{m_V^2}{E^2-m_V^2}\\&\times\int\frac{dE'^2(2E'^2-m_V^2)\mathrm{Im}\Pi(E')}{(E'^2-m_V^2)^2}\\&+\int\frac{dE'^2E'^4\mathrm{Im}\Pi(E')}{(E'^2-m_V^2)^2(E'^2-E^2)},\end{aligned} \tag{3.13}$$

which is the appropriate expression to study the neighborhood of the meson pole. If we now go to spacelike momenta and take the Borel transform, we get

$$\begin{aligned}\frac{1}{M^2}\int dE^2\mathrm{Im}\Pi(E)\Bigg[\Bigg\{&1-\left(\frac{E^2}{E^2-m_V^2}\right)^2\\&+\frac{m_V^2}{M^2}\frac{m_V^2}{E^2-m_V^2}\Bigg\}e^{-m_V^2/M^2}\\&+\left(\frac{E^2}{E^2-m_V^2}\right)^2e^{-E^2/M^2}\Bigg].\end{aligned} \tag{3.14}$$

Consider first the $\pi\omega$ self-energy loop, whose evaluation reveals the inadequacy of the old saturation scheme. For this loop we have $c=4(g_1/F_\pi)^2$ and the expressions for $f_{\mu\nu}$ and f are again given by Eqs. (3.6). Near the meson pole, the three integrals in Eq. (3.13) are all finite and are of order T^4, T^2 and T^2, respectively. Thus the pole position does not shift, but the residue does to order T^2 [15]. On the other hand, the Borel transform (3.14), which simplifies to leading order as

$$e^{-m_V^2/M^2}\int dE^2\left(1-\frac{E^2+m_V^2}{M^2}+\frac{E^4}{2M^4}\right)\mathrm{Im}\Pi(E),$$

is clearly of order T^4. To summarize, the self-energy diagram with the $\pi\omega$ loop gives a correction to the pole residue to order T^2, but the Borel transform of the full amplitude has no contribution to this order. By contrast, the old saturation scheme would retain the correction to the residue also in the Borel transform. It amounts to the neglect of the third, regular term in Eq. (3.13), which is of course justified near the pole, but not for large spacelike momenta, where the pole and the regular terms are not only of comparable magnitudes, but actually cancel each other in the leading order.

The πa_1 self-energy loop has

$$c=2(g_3/F_\pi)^2,\quad f_{\mu\nu}=B_{\mu\nu}-C_{\mu\nu}/m_A^2,$$

$$f=-|\vec{k}|^2(2+E^2/m_A^2)/3.$$

We see that this amplitude contributes neither to the pole position nor to the Borel transform to $O(T^2)$. The same is the case with the $\pi\pi$ loop as shown in the Appendix.

B. Axial-vector current

Here the Feynman diagrams are shown in Figs. 5–9. The new feature here is the existence of the pion pole in addition to the one for the axial vector meson a_1.

The π pole [Fig. 5(a)] is modified by the vertex corrections [Figs. 5(b) and 5(c)] and the self-energy correction [Fig. 9(a)]. Using the effective Lagrangians given by Eq. (2.13), the complete pole term in the chiral limit becomes

$$T'^{(\pi)}_{\mu\nu}=-F_\pi^2\left(1-\frac{T^2}{6F_\pi^2}\right)A_{\mu\nu}, \tag{3.15}$$

where we also include the contact diagrams of Figs. 5(d) and 5(e). The a_1 pole with vertex corrections of Fig. 6 is given by

$$T'^{(a_1)}_{\mu\nu}=-\left(\frac{F_{a_1}}{m_A}\right)^2\left(1-\frac{T^2}{6F_\pi^2}\right)\frac{q^4}{q^2-m_A^2}A_{\mu\nu}. \tag{3.16}$$

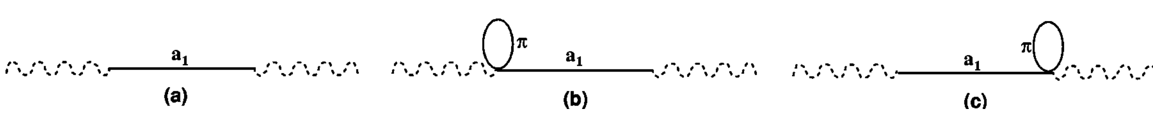

FIG. 6. a_1 pole and constant vertex corrections.

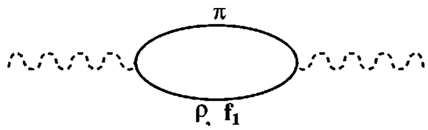

FIG. 7. Intermediate state diagrams.

The contribution of the $\pi\rho$ intermediate state of Fig. 7 to the invariant amplitudes $T_{l,t}$ are given by

$$T_{l,t}^{\prime(\pi\rho)}=\left(\frac{F_\rho}{m_V}\right)^2\frac{T^2}{6F_\pi^2}\frac{(-Q^2,Q^4)}{Q^2+m_V^2}. \tag{3.17}$$

As in the case of the vector current correlation function, one can show that none of the remaining diagrams contribute to order T^2 at large spacelike momenta.

IV. SUM RULES

We now turn to the operator side of the sum rules. Up to dimension six, the Lorentz scalar operators appearing in the short distance expansion of the product of two currents are **1**, $m_q\bar{q}q$, $G_{\mu\nu}G^{\mu\nu}$ and two four-quark operators [17]. Of the latter two, one arises from the Taylor expansion of the quark bilocal operators arising in the Wick expansion of the currents. This piece turns out to be isospin scalar. The other four-quark operator arises from the Feynman diagram to second order in QCD perturbation expansion, where the large external momentum flows through the internal gluon line [1].

In the medium there are additional operators, which are Lorentz non-scalars in the rest frame of the heat bath. They can, of course, be written as Lorentz scalars in a general frame by using the four-velocity vector u_μ. They first appear as dimension four, which are $\bar{q}\not{u}q$, $u^\mu u^\nu\Theta^f_{\mu\nu}$ and $u^\mu u^\nu\Theta^g_{\mu\nu}$, where $\Theta^{f,g}_{\mu\nu}$ are the energy momentum tensors of the quark and gluon, respectively. More such operators arise at dimensions five and six, but generally contain derivatives.

In the process of subtracting out the vacuum sum rule to extract terms of order T^2, the unit operator drops out. $m_q\bar{q}q$ also drops out in the chiral limit. The isospin scalar four-quark operator cannot have a temperature dependence in the chiral limit [3] . The operator $\bar{q}\not{u}q$ does not contribute, as we do not have the non-zero chemical potential for the fermions. The thermal expectation value of $G_{\mu\nu}G^{\mu\nu}$, $\Theta^f_{\mu\nu}$ and $\Theta^g_{\mu\nu}$ are all of order T^4 [4]. Also the expectation values of other operators with derivative(s) are of higher order than T^2, as each derivative gives rise to an additional power in pion momentum in their pion matrix elements.

Thus we are left with only the isospin non-scalar four quark operator as the relevant one for our sum rules. Their Wilson coefficients are known [1],

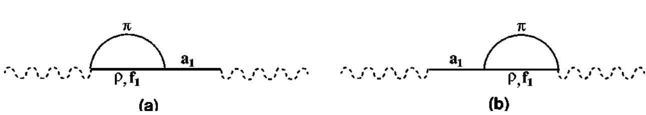

FIG. 8. Vertex correction diagrams.

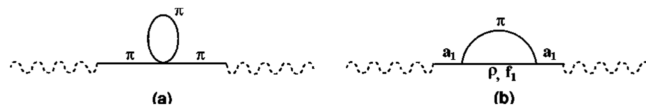

FIG. 9. Self-energy diagrams.

$$i\int d^4x\, e^{iq\cdot x}\langle T\, V^a_\mu(x)V^b_\nu(0)\rangle$$
$$\to\delta^{ab}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{8\pi}{3Q^4}\langle O_A\rangle,$$
$$i\int d^4x\, e^{iq\cdot x}\langle T\, A^a_\mu(x)A^b_\nu(0)\rangle$$
$$\to\delta^{ab}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{8\pi}{3Q^4}\langle O_V\rangle, \tag{4.1}$$

where

$$O_V=\alpha_s\bar{q}\gamma_\mu\frac{\tau^c}{2}\frac{t^i}{2}q\bar{q}\gamma^\mu\frac{\tau^c}{2}\frac{t^i}{2}q,$$

$$O_A=\alpha_s\bar{q}\gamma_\mu\gamma_5\frac{\tau^c}{2}\frac{t^i}{2}q\bar{q}\gamma^\mu\gamma_5\frac{\tau^c}{2}\frac{t^i}{2}q,$$

and $\alpha_s=g_s^2/4\pi$ is the strong interaction fine structure constant.

The thermal average of any operator may be expanded as

$$\langle O\rangle=\langle 0|O|0\rangle+\sum_a\int\frac{d^3kn(k)}{(2\pi)^3 2\omega_k}\langle\pi^a(k)|O|\pi^a(k)\rangle+\cdots, \tag{4.2}$$

where the sum is over the isospin indices of the pion. The pion matrix elements of $O_{V,A}$ are easily worked out using PCAC (partially conserved axial-vector current) and current algebra to get

$$\langle O_{V,A}\rangle=\langle 0|O_{V,A}|0\rangle\mp\frac{T^2}{6F_\pi^2}\langle 0|O_V-O_A|0\rangle.$$

It is now simple to write the sum rules for a correlation function by equating the Borel transform of the spectral representation to the operator product expansion. It turns out that both the vector and the axial vector correlation functions give rise to the same sum rules [18]:

$$-F_\rho^2 e^{-m_V^2/M^2}+F_{a_1}^2 e^{-m_A^2/M^2}+F_\pi^2=-\frac{4\pi}{3M^4}\langle 0|O_V-O_A|0\rangle,$$

$$-m_V^2F_\rho^2 e^{-m_V^2/M^2}+m_A^2F_{a_1}^2 e^{-m_A^2/M^2}=\frac{8\pi}{3M^2}\langle 0|O_V-O_A|0\rangle. \tag{4.3}$$

It is interesting to observe that these sum rules are nothing but the vacuum sum rules derived from the difference of the correlation functions of the vector and the axial-vector currents, whose spectral side consists of the pole terms due to π, ρ and a_1 exchanges. Also as M^2 tends to infinity, we recover the well known sum rules

$$F_\rho^2 - F_{a_1}^2 - F_\pi^2 = 0,$$

$$m_V^2 F_\rho^2 - m_A^2 F_{a_1}^2 = 0, \tag{4.4}$$

originally derived by Weinberg [19] as superconvergence sum rules for the difference of the spectral functions.

V. CONCLUDING REMARKS

In this work we have obtained the sum rules following from the two point functions of the vector current and the axial vector current at finite temperature. For a reliable estimate, we calculate their spectral sides using the chiral Lagrangian. The resulting Feynman diagrams (to one loop) can be readily evaluated for their leading thermal contributions.

Though there is a multitude of diagrams to begin with, only a few of these actually contribute to order T^2. This is due to the presence of derivatives on the pion fields in the interaction vertices as required by chiral symmetry. Among the diagrams, there is a one-particle reducible, self-energy diagram, whose contribution at spacelike momenta is zero to order T^2, but the old saturation scheme would derive from this diagram a shift to this order in the residue of the meson pole.

The older strategy of determining the temperature dependence of the pole parameters from such sum rules is not relevant anymore, as we have already included the diagrams responsible for this dependence in our calculation of the spectral functions. Our evaluation reproduces the sum rules which follow from the vacuum correlation functions themselves.

The present saturation scheme has been used also for the nucleon correlation function at finite temperature, again reproducing the results obtainable from the corresponding vacuum correlation function [6,7]. Thus we now have enough confirmation of the correctness of the QCD sum rules in a medium.

Can we get *new* results from the sum rules in a medium? It would seem that we are merely reproducing the old results in a more complicated way: What one could have from single particle exchange diagrams in vacuum are now derived from one-loop diagrams at finite temperature. But we do not believe the situation to be always so. Indeed, having been assured of the correct procedure to saturate the spectral side, we may apply the sum rule technique to other media, such as the nuclear medium, by introducing the nucleon chemical potential. There are no reliable estimates of higher dimension operators like the four-quark operators in such media. It is these quantities which should be readily available from such sum rules.

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APPENDIX

Some care is necessary in constructing the spectral representation for the $\pi\pi$ loop for $\vec{q}=0$. The loop integral gives rise to cuts in the q_0^2 plane for arbitrary $\vec{q}$ extending over $0<q_0^2<|\vec{q}|^2$ (short cut) and $q_0^2>4(m_\pi^2+|\vec{q}|^2)$ (unitary cut). As $\vec{q}\rightarrow 0$ the short cut shrinks to a point. But the spectral function can be singular in this limit, so that the part of the spectral representation due to this cut may be non-vanishing.

Consider first the $\pi\pi$ loop in the vertex diagram. It gives an amplitude of the form of Eq. (3.7), where

$$\Lambda_{\mu\nu}(q) = ic\int \frac{d^4k}{(2\pi)^4} f_{\mu\nu}(q,k)\Delta_{11}^{(\pi)}(k)\Delta_{11}^{(\pi)}(q-k), \tag{A1}$$

with

$$c=(4G_\rho F_\rho/m_V^2F_\pi^2),$$

$$f_{\mu\nu}=(q_\mu-2k_\mu)(q^2k_\nu-q\cdot kq_\nu).$$

As in Sec. II the imaginary part of the amplitude may again be obtained directly by integrating over k_0. But, unlike the case of loops with one pion and one heavy particle, the angular integration in the resulting integral over $\vec{k}$ gives rise to a θ-function constraining the limits of $|\vec{k}|$. The imaginary part of the invariant amplitudes may then be worked out to give

$$\begin{pmatrix}\mathrm{Im}\Lambda_l\\ \mathrm{Im}\Lambda_t\end{pmatrix} = \frac{1}{8(2\pi)^2}\int_{-1}^{+1}dx\begin{pmatrix}-q^2x^2/2\\ -q^4/6\end{pmatrix} \times\left(1+\frac{2}{e^{\beta(|\vec{q}|x+q_0)/2}-1}\right),\quad q^2>0, \tag{A2}$$

and

$$\begin{pmatrix}\mathrm{Im}\Lambda_l\\ \mathrm{Im}\Lambda_t\end{pmatrix} = \frac{1}{8(2\pi)^2}\int_{1}^{\infty}dx\begin{pmatrix}-q^2x^2/2\\ -q^4/6\end{pmatrix} \times\left(\frac{1}{e^{\beta(|\vec{q}|x-q_0)/2}-1}-\frac{1}{e^{\beta(|\vec{q}|x+q_0)/2}-1}\right),\quad q^2<0. \tag{A3}$$

On the unitary cut the x integrals are thus finite as $\vec{q}\rightarrow 0$ and because of $q^2(q^4)$ in the integrands, their Borel transforms will have no contributions to $O(T^2)$. But on the short cut the x integrals diverge as $\vec{q}\rightarrow 0$. Let us then consider the full Borel transformed amplitude with $\vec{q}\neq 0$,

$$\begin{pmatrix}\Lambda_l\\ \Lambda_t\end{pmatrix}=\frac{1}{M^2}\int_0^{|\vec{q}|^2}dq_0^2 e^{-q_0^2/M^2}\begin{pmatrix}\mathrm{Im}\Lambda_l\\ \mathrm{Im}\Lambda_t\end{pmatrix}. \tag{A4}$$

Defining new variables λ and u by $q_0=\lambda|\vec{q}|$ and $|\vec{q}|x=u$, it becomes

$$\begin{pmatrix}\Lambda_l\\ \Lambda_t\end{pmatrix}=\frac{|\vec{q}|^2}{8(2\pi)^2M^2}\int_0^1 d\lambda^2 e^{-\lambda^2|\vec{q}|^2/M^2}\lambda \times\int_{|\vec{q}|}^{\infty}du\begin{pmatrix}(1-\lambda^2)u^2/2\\ -(1-\lambda^2)^2|\vec{q}|^4/6\end{pmatrix}\cdot D \tag{A5}$$

where

$$D=\frac{1}{\lambda|\vec{q}|}\left(\frac{1}{e^{\beta(u-\lambda|\vec{q}|)/2}-1}-\frac{1}{e^{\beta(u+\lambda|\vec{q}|)/2}-1}\right) \rightarrow -2\frac{d}{du}\left(\frac{1}{e^{\beta u/2}-1}\right), \tag{A6}$$

as $|\vec{q}|\rightarrow 0$. Thus the short cut also cannot contribute for $\vec{q}=0$.

The $\pi\pi$ loop in the self-energy correction is given by Eq. (3.12) where $\Pi_{\mu\nu}$ is of the form of Eq. (A1) with $c=(2G_\rho/m_V F_\pi^2)^2$ and $f_{\mu\nu}=C_{\mu\nu}$. From the previous treatment it is clear that this amplitude again does not lead to any corrections of $O(T^2)$.

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